# INSTRUMENTAL VARIABLE ESTIMATION OF THE SIMPLE ERRORS IN VARIABLES MODEL

R. L. Carter, Florida University
W. A. Fuller, Iowa State University

ABSTRACT

Alternative instrumental variable estimators for the slope in the simple errors in variables model are discussed. The modified limited information estimator is used to construct a randomly weighted average estimator similar to those studied by Huntsberger. The maximum likelihood estimator is derived for the case in which the error covariance is known to be zero. The limiting distributions of these estimators are obtained. The modified maximum likelihood estimators, the randomly weighted average estimators and Feldstein's randomly weighted average estimator are compared in a Monte Carlo study.

## 1. INTRODUCTION

The errors in variables model differs from the classical linear regression model in that the independent variables are not observed directly, but are masked by measurement error. For such models it is well known that the ordinary least squares estimators are biased. Generally speaking, estimation of the parameters of such a model requires additional information. This information may take the form of knowledge of the error variances, observations on other variables which are uncorrelated with the measurement errors, or knowledge of the form of the distribution of the errors and (or) regression variables. The case in which observations on additional variables, called instrumental variables, are available is considered in this paper. For discussions of the other two cases see [2], [6], [12] and [13].

The use of instrumental variables to estimate the parameters of errors in variables models has been heavily used in economics. Reiersol [14], Geary [7], Durbin [3] and Sargan [15] have discussed this method and most modern econometric texts contain a discussion of instrumental variable estimation. See, for example, Johnston [11] and Goldberger [8].

The methods used to estimate a single equation in a system of equations are closely related to the method of instrumental variables. In particular, for a just identified equation the two stage least squares and limited information maximum likelihood procedures reduce to the instrumental variable procedure. It is well known that these procedures yield estimators that may not possess moments in finite samples. Fuller [5] gave a modification of the limited information estimator and derived its mean square error through terms of order $T^{-2}$.

Huntsberger [10] investigated estimators of the mean of the normal population, constructed as randomly weighted averages of two estimators. One estimator, denoted by $\hat{\theta}_1$, is unbiased for the mean, $\theta_1$, and has variance $A_1$. The other estimator, $\hat{\theta}_2$, is independent of $\hat{\theta}_1$ and has an unknown mean, $\theta_2$, and variance $A_2$. We denote Huntsberger's family of estimators by

$$\hat{\theta}_H = \hat{a}\,\hat{\theta}_1 + (1-\hat{a})\hat{\theta}_2, \tag{1.1}$$

where the weight, $\hat{a}$ is a function of the sample. The weight

$$\hat{a}_w = \frac{(\hat{\theta}_1 - \hat{\theta}_2)^2 + A_2}{A_1 + A_2 + (\hat{\theta}_1 - \hat{\theta}_2)^2} \tag{1.2}$$

is an estimator of the fixed weight that minimizes the mean square error of a fixed weight estimator of the form (1.1). We call an estimator of this type a randomly weighted estimator. Huntsberger demonstrated that the mean square error of $\hat{\theta}_H$ is a function of $\gamma$, where

$$\gamma = (A_1 + A_2)^{-\frac{1}{2}}\,|\theta_1 - \theta_2| \ . \tag{1.3}$$

The mean square error of $\hat{\theta}_H$ is less than that of $\hat{\theta}_1$ if $\gamma=0$. As $\gamma$ increases the mean square error rises to a maximum above the mean square error of $\hat{\theta}_1$. If $A_1=A_2$ and the estimator with $\hat{a}_w$ is used the maximum occurs near $\gamma=2.75$. As $\gamma$ increases the mean square error of $\hat{\theta}_H$ approaches that of $\hat{\theta}_1$. Changing the form of $\hat{a}$ alters the mean square error function somewhat, but in all cases the function has the distinctive shape described above.

Feldstein [4] recently proposed an estimator for the measurement error model with an instrumental variable present. The estimator is an average of the instrumental variable and ordinary least squares estimators similar to (1.1). Therefore, one might expect the behavior of the estimator to depend in a critical manner on the true parameters. Feldstein, however, presented a Monte Carlo study showing his estimator to be uniformly superior to the instrumental variable estimator.

For the errors in variables model with one independent variable we compare Fuller's estimator with randomly weighted average estimators of the type considered by Huntsberger and Feldstein. We shall identify two models which result in slightly different estimation procedures.

## 2. THE MODELS

Let the errors in variables model be defined by

$$\underset{\sim}{Y} = \underset{\sim}{x}\beta + \underset{\sim}{e}\ , \qquad \underset{\sim}{X} = \underset{\sim}{x} + \underset{\sim}{u}\ , \tag{2.1}$$

where

- $\underset{\sim}{Y}$ is a Tx1 vector of observations on the dependent variable Y,
- $\underset{\sim}{x}$ is a Tx1 vector of unobservable true values of the independent variable x,
- $\underset{\sim}{e}$ is a Tx1 vector of errors,
- $\underset{\sim}{X}$ is a Tx1 vector of observations on the observable random variable X,
- $\underset{\sim}{u}$ is a Tx1 vector of measurement errors.

Rewriting the model in terms of observable random variables we have:

$$\underset{\sim}{Y} = \underset{\sim}{X}\beta + \underset{\sim}{w},\ \text{where}\ \underset{\sim}{w} = \underset{\sim}{e} - \underset{\sim}{u}\beta\ .$$

We assume the existence of a Tx1 vector of observations on an instrumental variable, $\underset{\sim}{z}$, with $t^{th}$ observation denoted by $z_t$. It is also assumed that $[e_t, u_t, x_t, z_t]'$, $t=1,2,\ldots,T$, are distributed as independent drawings from a multivariate normal distribution with mean $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{\Sigma}$, where

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \sigma_e^2 & \sigma_{ue} & 0 & 0 \\ \sigma_{ue} & \sigma_u^2 & 0 & 0 \\ 0 & 0 & \sigma_x^2 & \sigma_{xz} \\ 0 & 0 & \sigma_{xz} & \sigma_z^2 \end{pmatrix}, \tag{2.3}$$

$\sigma_{xz} \neq 0$ and $\sigma_x^2 > 0$. We note that the model as defined is equivalent to the system of structural equations

$$\begin{aligned} Y_t &= \beta X_t + w_t \\ X_t &= \delta z_t + \epsilon_t , \end{aligned} \tag{2.4}$$

where $Y_t$ and $X_t$ are endogenous variables, $z_t$ is an exogenous variable, and $\delta = \sigma_{xz}/\sigma_z^2$.

The model is easily generalized to contain an intercept term and the assumption of zero means for all variables is an assumption of convenience. The assumption of normality is stronger than necessary for many of the results used in this paper, but simplifies the presentation and is the model used in our Monte Carlo study and that of Feldstein [4].

Under certain circumstances it may be known that $\sigma_{ue}=0$. This situation is identified as model b). The situation wherein $\sigma_{ue}$ is unknown and unrestricted is model a).

## 3. ESTIMATION

### 3.1. Estimation when $\sigma_{ue}$ is unknown (model a)

The reduced form associated with (2.4) is given by

$$\begin{aligned} Y_t &= \beta\delta z_t + \xi_t , \\ X_t &= \delta z_t + \epsilon_t , \end{aligned} \tag{3.1}$$

where $\xi_t$ and $\epsilon_t$ are the reduced form errors. We assume $\sigma_\epsilon^2 > 0$ for model a). For this model the maximum likelihood estimator of $\beta$ is the instrumental variable estimator,[1]

$$\tilde{\beta}_{IV} = \Big(\sum_{t=1}^{T} X_t z_t\Big)^{-1} \sum_{t=1}^{T} Y_t z_t . \tag{3.2}$$

Under the model assumptions, it is well known that

$$T^{1/2}(\tilde{\beta}_{IV}-\beta) \xrightarrow{\mathcal{L}} N(0,[\delta^2\sigma_z^2]^{-1}\sigma_w^2) . \tag{3.3}$$

See, for example, Fuller [5].

The matrix of sums of squares and cross products of residuals from the reduced form regressions is

$$\underset{\sim}{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{pmatrix} ,$$

where

$$W_{11} = \Sigma\hat{\xi}^2 = \Sigma Y^2 - [\Sigma z^2]^{-1}(\Sigma Yz)^2 , \tag{3.5}$$

$$W_{12} = \Sigma\hat{\xi}\hat{\epsilon} = \Sigma XY - [\Sigma z^2]^{-1}(\Sigma Xz)(\Sigma Yz) \tag{3.6}$$

$$W_{22} = \Sigma\hat{\epsilon}^2 = \Sigma X^2 - [\Sigma z^2]^{-1}(\Sigma Xz)^2 . \tag{3.7}$$

The family of modified limited information maximum likelihood estimators of $\beta$ considered by Fuller [5] is defined by

$$\tilde{\beta}_{ML\alpha} = \frac{(T-1)(\Sigma Yz)(\Sigma Xz) + \alpha W_{12}(\Sigma z^2)}{(T-1)(\Sigma Xz)^2 + \alpha W_{22}(\Sigma z^2)} , \tag{3.8}$$

where $\alpha$ is a fixed positive real number.

The ordinary least squares estimator of $\beta$ is $\hat{\beta}_{OLS} = (\Sigma X^2)^{-1}\Sigma XY$. When using the mean square error as the basis of comparison, neither the ordinary least squares nor the instrumental variable estimator is preferred for all parameter values.

We now construct an estimator for $\beta$ of the form (1.1) with weights (1.2). In our construction $\tilde{\beta}_{ML4}$ plays the role of $\theta_1$ and $\tilde{\beta}_R = W_{22}^{-1}W_{12}$, where $W_{22}$ and $W_{12}$ are defined in (3.7) and (3.6), plays the role of $\hat{\theta}_2$. Our randomly weighted average estimator is

$$\tilde{\beta}_W = \tilde{a}_W\tilde{\beta}_{ML4} + (1-\tilde{a}_W)\tilde{\beta}_R, \tag{3.15}$$

where

$$\tilde{a}_W = \frac{(\tilde{\beta}_{ML4} - \tilde{\beta}_R)^2 + \hat{A}_2}{\hat{A}_1 + \hat{A}_2 + (\tilde{\beta}_{ML4} - \tilde{\beta}_R)^2} ,$$

$$\hat{A}_2 = \frac{(T-1)S_1^2}{(T-5)W_{22}} ,$$

$$S_1^2 = (T-2)^{-1}(W_{11} - W_{12}\tilde{\beta}_R) .$$

The following theorem demonstrates that $T^{1/2}(\tilde{\beta}_W-\beta)$ converges in distribution to the estimator studied by Huntsberger.

<u>Theorem 3.1.1</u>. Let $H(\theta,\gamma,A_1,A_2)$ denote the distribution of the randomly weighted average estimator with weight (1.2).

Let $(X,Y,z)$ be distributed as a trivariate normal random variable satisfying model (2.1), (2.2), (2.3), (2.4) with $\delta \neq 0$ and $\sigma_x^2 - \delta^2\sigma_z^2 > 0$. Let all parameters except $\sigma_u^2$ and $\sigma_{ue}$ be fixed. Let $\sigma_u^2 = \alpha T^{-1/2}$ and $\sigma_{ue} = \eta\sigma_u^2$ where $\alpha$ and $\eta$ are fixed. Let $\tilde{\beta}_W$ be defined by (3.15). Then

$$T^{1/2}(\tilde{\beta}_W-\beta) \xrightarrow{\mathcal{L}} H(0,\gamma,A_1,A_2) ,$$

where

$$\gamma^2 = \frac{\alpha^2(\eta-\beta)^2\delta^2\sigma_z^2}{\sigma_e^2\sigma_x^2(\sigma_x^2-\delta^2\sigma_z^2)}$$

$$A_1 = (\delta^2\sigma_z^2)^{-1}\sigma_e^2$$

[1] All summations in this paper are over t as t ranges from 1 to T. Henceforth, we shall suppress the subscripts and the range of summation.

$$A_2 = (\sigma_x^2 - \delta^2\sigma_z^2)^{-1}\sigma_e^2 .$$

3.2. Estimation when $\sigma_{ue}$ is known (model b)

By assumption, the instrumental variable z is correlated with x and uncorrelated with u and e. Thus, we may write

$$z_t = \rho x_t + v_t , \qquad (3.18)$$

where $v_t$, t = 1,2,...,T are normal independent $(0,\sigma_v^2)$ random variables independent of $u_j, e_j$ and $x_j$ for all t and j. Given our model, with the added assumption that $\sigma_{ue} = 0$, the vector $(Y_t, X_t, z_t)$ is normally distributed with zero mean and covariance matrix $\underset{\sim}{V}$, where

$$\underset{\sim}{V} = \begin{pmatrix} \beta^2\sigma_x^2+\sigma_e^2 & \beta\sigma_x^2 & \rho\beta\sigma_x^2 \\ \beta\sigma_x^2 & \sigma_x^2+\sigma_u^2 & \rho\sigma_x^2 \\ \rho\beta\sigma_x^2 & \rho\sigma_x^2 & \rho^2\sigma_x^2+\sigma_v^2 \end{pmatrix} . \qquad (3.19)$$

We shall obtain the maximum likelihood estimator of $\underset{\sim}{\theta} = (\beta, \sigma_x^2, \rho, \sigma_e^2, \sigma_u^2, \sigma_v^2)'$ under the assumptions; $\beta \neq 0$, $\rho \neq 0$, $\sigma_x^2 \neq 0$, $\sigma_e^2 \geq 0$, $\sigma_u^2 \geq 0$, $\sigma_v^2 \geq 0$. The inequality restrictions on $\beta$, $\rho$ and $\sigma_x^2$ are required for all parameters of the model to be identified. Once the maximum likelihood estimator of $\underset{\sim}{\theta}$ has been obtained we shall demonstrate that, given the remaining assumptions, the estimator of $\beta$ is consistent for all $\beta$, including $\beta = 0$. We also temporarily assume that at most one of the variances $\sigma_e^2$, $\sigma_u^2$ or $\sigma_v^2$ is zero. If two or more of the population error variances are zero, then the matrix $\underset{\sim}{V}$, defined in (3.19), is singular and the vector $(Y,X,z)'$ has a singular normal distribution. This situation is easily detected in the sample because the matrix of sums of squares and cross-products of $(Y,X,z)'$ is singular. Therefore, this case will be treated separately. Under the present assumptions, the space of admissible values for the parameter vector, $\underset{\sim}{\theta}$, is denoted by $\Theta$.

If there are no restrictions on the matrix $\underset{\sim}{V}$, the maximum likelihood estimators of the elements of $\underset{\sim}{V}$ are given by the sample moments, $\hat{V}_{11} = s_X^2 = T^{-1}\Sigma X^2$, $\hat{V}_{22} = s_Y^2 = T^{-1}\Sigma Y^2$, $\hat{V}_{12} = s_{XY} = T^{-1}\Sigma XY$, $\hat{V}_{13} = s_{Yz} = T^{-1}\Sigma Yz$ and $\hat{V}_{23} = s_{Xz} = T^{-1}\Sigma Xz$ (see [1], Chap. 3). The <u>simple</u> estimator of the parameter vector, $\underset{\sim}{\theta}$, is obtained by solving the equations $\underset{\sim}{V} = \underset{\sim}{\hat{V}}$ for the parameters of interest. The following theorem defines the maximum likelihood estimator of $\underset{\sim}{\theta}$ (i.e. the estimator that maximizes the likelihood on the parameter space $\Theta$).

<u>Theorem 3.2.1</u>. Let (Y,X,z) be distributed as a trivariate normal random variable with mean zero and covariance matrix $\underset{\sim}{V}$, defined in (3.19). Let the parameter $\underset{\sim}{\theta} \in \Theta$ and let G denote the event that the simple estimator does not lie in $\Theta$. Let $\hat{R}_{XY}^2 = (s_Y^2 s_X^2)^{-1} s_{XY}^2$, $\hat{R}_{Xz}^2 = (s_X^2 s_z^2)^{-1} s_{Xz}^2$, $\hat{R}_{Yz}^2 = (s_Y^2 s_z^2)^{-1} s_{Yz}^2$ and $R_{min}^2 = \min\{\hat{R}_{XY}^2, \hat{R}_{Xz}^2, \hat{R}_{Yz}^2\}$ and define

$$\hat{\underset{\sim}{\theta}}_e = (\hat{\beta}_{IN}, s_Y^{-2}s_{XY}^2, s_{XY}^{-1}s_{Yz}, 0, s_X^2 - s_Y^{-2}s_{XY}^2, s_z^2 - s_Y^{-2}s_{Yz}^2)' ,$$
$$\hat{\underset{\sim}{\theta}}_u = (\hat{\beta}_{OLS}, s_X^2, s_X^{-2}s_{Xz}, s_Y^2 - s_X^{-2}s_{XY}^2, 0, s_z^2 - s_X^{-2}s_{Xz}^2)' ,$$
$$\hat{\underset{\sim}{\theta}}_v = (\tilde{\beta}_{IV}, s_z^{-2}s_{Xz}^2, s_{Xz}^{-1}s_z^2, s_Y^2 - s_z^{-2}s_{Yz}^2, s_X^2 - s_z^{-2}s_{Xz}^2, 0)' ,$$
$$\hat{\beta}_{IN} = s_{XY}^{-1}s_Y^2 . \qquad (3.29)$$

Then the maximum likelihood estimator of $\underset{\sim}{\theta}$ is

$$\hat{\theta}_M = \begin{cases} \hat{\underset{\sim}{\theta}}_e, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Xz}^2 \\ \hat{\underset{\sim}{\theta}}_u, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Yz}^2 \\ \hat{\underset{\sim}{\theta}}_v, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{XY}^2 \\ \hat{\underset{\sim}{\theta}} , & \text{otherwise.} \end{cases} \qquad (3.30)$$

That is, $\hat{\underset{\sim}{\theta}}_M \in \Theta$ and $L(\hat{\theta}_M) \geq L(\underset{\sim}{\theta}^+)$ for all $\underset{\sim}{\theta}^+ \in \Theta$.

If exactly two of the variances are zero, then there is a perfect correlation between the two variables with no measurement error. In such a case the model can be reduced to a regression model in one of these variables and the third variable. The maximum likelihood estimator for the slope parameter of the reduced model is obtained by ordinary least squares. The maximum likelihood estimator was constructed under the assumption that the parameter $\beta$ is not zero. It can be proven that the first component of $\tilde{\underset{\sim}{\theta}}_{ML}$ is consistent for $\beta$ when $\beta = 0$.

Recall that the modified estimator, $\tilde{\beta}_{ML4}$, has moment properties superior to those of $\tilde{\beta}_{IV}$. The estimator, $\hat{\beta}_{IN}$, defined in (3.29) can be modified in the same way to produce the estimator,

$$\hat{\beta}_{IN4} = \frac{(T-1)\Sigma XY\Sigma Y^2}{(T-5)(\Sigma XY)^2 + 4\Sigma Y^2\Sigma X^2} ,$$

with moment properties superior to $\hat{\beta}_{IN}$. On the basis of these results and Theorem 3.2.1, we propose the following estimator for $\beta$ when $\sigma_{ue}$ is known to be zero:

$$\tilde{\beta}_{MA} = \begin{cases} \hat{\beta}_{OLS}, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Yz}^2, \text{ or } \sigma_u^2 = \sigma_e^2 = 0, \\ & \text{or } \sigma_u^2 = \sigma_v^2 = 0 \\ \hat{\beta}_{IN4}, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Xz}^2, \text{ or } \sigma_e^2 = \sigma_v^2 = 0, \\ & \sigma_u^2 \neq 0 \\ \tilde{\beta}_{ML4}, & \text{otherwise.} \end{cases} \qquad (3.36)$$

A weighted average estimator analogous to (3.36) is

$$\tilde{\beta}_{WA} = \begin{cases} \hat{\beta}_{OLS}, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Yz}^2, \text{ or } \sigma_u^2 = \sigma_e^2 = 0, \\ & \text{or } \sigma_u^2 = \sigma_v^2 = 0 \\ \hat{\beta}_{IN4}, & \text{if } G \text{ and } R_{min}^2 = \hat{R}_{Xz}^2, \text{ or } \sigma_e^2 = \sigma_v^2 = 0, \\ & \sigma_u^2 \neq 0 \\ \tilde{\beta}_W , & \text{otherwise,} \end{cases} \qquad (3.39)$$

where $\tilde{\beta}_W$ is defined in (3.15).

An estimator patterned after the arguments of Feldstein[1] is

$$\tilde{\beta}_K = \hat{\eta}\,\hat{\beta}_{OLS} + (1-\hat{\eta})\tilde{\beta}_{IV} \tag{3.40}$$

where

$$\hat{\eta} = [1+(T+1)\hat{K}^2\{H_1(\hat{K}^2+H_2)+2\hat{K}^2\}^{-1}]^{-1}$$

$$H_1 = s_{Xz}^{-2}(s_X^2 s_z^2 - s_{Xz}^2)$$

$$H_2 = s_{XY}^{-2}(s_X^2 s_Y^2 - s_{XY}^2)$$

$$\hat{K} = \begin{cases} (\hat{\beta}_{OLS})^{-1}\,\tilde{\beta}_{IV}-1 & \text{if } (\hat{\beta}_{OLS})^{-1}\,\tilde{\beta}_{IV} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Feldstein defined $\hat{K}$ to be an estimator of $K = \sigma_x^{-2}\sigma_u^2$. For details the reader is referred to [4].

## 4. EMPIRICAL RESULTS

Simulation experiments were carried out to compare the estimators $\hat{\beta}_{OLS}$, $\tilde{\beta}_{ML1}$, $\tilde{\beta}_{ML4}$, $\tilde{\beta}_W$, $\tilde{\beta}_{MA}$, $\tilde{\beta}_{WA}$ and $\tilde{\beta}_K$. Our study follows closely that of Feldstein [4]. As in that study we considered two cases, T = 25 and T = 100, with $\beta = 1$, $\sigma_{ue} = 0$ and $\sigma_x^2 = \sigma_z^2 = 100$. The remaining parameters, $\sigma_u^2$, $\sigma_e^2$ and $\sigma_{xz}$, were also chosen to coincide with Feldstein's study. Two thousand samples were generated and the seven estimators were computed. The random number package, "Super Duper," from McGill University was used to generate values for the normal random variables $e_t$, $u_t$, $x_t$ and $z_t$.

Three conclusions are possible from the Monte Carlo results for estimators that do not use the information $\sigma_{ue} = 0$. These are:

1) The modified limited information estimator with $\alpha=4$ has mean square error which is smaller than that with $\alpha=1$ for all parameter sets.
2) The modified limited information estimator with $\alpha=1$ is very nearly unbiased for all values of the parameters.
3) The estimator $\tilde{\beta}_W$ behaves like the randomly weighted average estimator discussed by Huntsberger. Loosely speaking, the mean square error of $\tilde{\beta}_{ML4}$ is larger than that of $\tilde{\beta}_W$ when $\gamma < 1.17$. As $\gamma$ increases the relative mean square error of $\tilde{\beta}_W$ reaches a maximum of about 1.2 for $\gamma$ between 2.0 and 3.0. As $\gamma$ becomes larger the mean square error of $\tilde{\beta}_W$ approaches that of $\tilde{\beta}_{ML4}$.

The superiority of $\tilde{\beta}_{ML4}$ over $\tilde{\beta}_{ML1}$ was greater for the smaller correlation between X and z. This was expected because the two estimators are identical if $R_{Xz} = 1$.

Similar conclusions are reached when the estimators using the information that $\sigma_{ue}=0$ are compared to that of $\tilde{\beta}_{MA}$. The ratio of the mean square error of $\tilde{\beta}_{WA}$ is less than one for $\gamma=0$, increases to a peak above one and then approaches one from above as $\gamma$ increases. Therefore, the comments made about the pair $\tilde{\beta}_{ML4}$ and $\tilde{\beta}_W$ also apply to the pair $\tilde{\beta}_{MLA}$ and $\tilde{\beta}_{WA}$. The mean square error of $\tilde{\beta}_K$ is uniformly larger than that of $\tilde{\beta}_{WA}$. The superiority of $\tilde{\beta}_{WA}$ can be attributed to two factors. First, $\tilde{\beta}_{ML4}$ performs uniformly better than the instrumental variable estimator, $\tilde{\beta}_{IV}$. Thus, a randomly weighted average of $\tilde{\beta}_{ML4}$ and $\tilde{\beta}_{OLS}$ is expected to perform better than a randomly weighted average of $\tilde{\beta}_{IV}$ and $\hat{\beta}_{OLS}$. Second, $\tilde{\beta}_{WA}$ is restricted so that the estimators for $\sigma_x^2$, $\sigma_u^2$, $\sigma_e^2$ and $\sigma_v^2$ are nonnegative whereas $\tilde{\beta}_K$ only guarantees a nonnegative estimator for $\sigma_x^{-2}\sigma_u^2$.

When the estimators using the information $\sigma_{ue}=0$ are compared to those that do not, the estimators using the information are generally superior.

## 5. CONCLUSION

On the basis of this study Fuller's modified limited information estimator with $\alpha=4$ can be recommended for instrumental variable estimation when $\sigma_{ue}$ is unknown and the objective is to minimize the mean square error of the estimator of $\beta$. Likewise, the adjusted maximum likelihood estimator, $\tilde{\beta}_{MA}$, can be recommended if $\sigma_{ue}$ is known to be zero. The mean square error function of the randomly weighted estimators is similar to that obtained by Huntsberger for analogous combinations of normal estimators. That is, the randomly weighted estimator has a smaller mean square error than the modified maximum likelihood estimator if the bias in the second estimator is 'small' and a larger mean square error otherwise.

Proof of the theorems and tables of the Monte Carlo results are contained in a larger manuscript which can be obtained from the authors.

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[1]The estimator is not identical to that considered by Feldstein.

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